The stability diagram of Eq. (44) is shown in Fig. 6. The larger the mass of the astronaut and the farther he moves, the larger will be the value of q. When q is very small, instability may occur if the man moves periodically along a radius with a period that is an integral multiple of the half-period of the spin of the satellite. If the man's weight is larger, or if his amplitude of motion increases, the unstable regime will be wider.

If the other coefficients θ_0, θ_{1s}, θ_{2c}, θ_{2s}, . . . in Eq. (43) cannot be ignored, we must deal with Hill's equation (43). The stability diagram is fairly similar to Fig. 6.

Conclusions

We have shown that, if a man on a spinning space station moves periodically, he can rock the station and cause it to tumble. The crucial parameter, of course, is the period of his motion. Usually he should avoid having a periodic motion with a period in the neighborhood of an integral multiple of the half-period of the spin of the satellite. The exact forbidden intervals of the periods of motion depend on the type and amplitude of his motion, the satellite's geometry and inertia, the man's mass, etc. Several examples are given in this paper.

References

[1] Hayashi, C., *Forced Oscillations in Non-Linear Systems* (Nippon Publishing Co., Osaka, Japan, 1953).

[2] *Tables Relating to Mathieu Functions*, National Bureau of Standards, Applied Mathematics Labs. (Columbia University Press, New York, 1951).

[3] McLachlan, N. W., *Theory and Application of Mathieu Functions* (Clarendon Press, Oxford, England, 1947).

JUNE 1965 AIAA JOURNAL VOL. 3, NO. 6

Tumbling Motions of an Artificial Satellite

Ralph Pringle Jr.*

Lockheed Missiles and Space Company, Palo Alto, Calif.

This paper treats the motion of a passively damped, gravity stabilized, artificial satellite that is tumbling or rotating about an arbitrary axis. This type of motion may occur after launch and separation, but before "capture" into a libration motion. A perturbation method is applied to this nonlinear problem. It is shown that the gravity gradient torques cause internal friction with consequent decrease of the body angular rates. Results are obtained for decay of the tumbling rate from an initial rate to the beginning of bounded libration. It is found that, for tumbling angular rates greater than three times the mean orbit angular rate, the time to capture increases as the cube of the initial rate.

Nomenclature

$(d/d\tau)(\ )$ = $d/d(nt)(\ ) = (\ )'$
n = orbital angular rate
τ = nt = normalized time
ϕ, θ, γ = Euler angles of the body axes
$\hat{1}, \hat{2}, \hat{3}$ = orbital reference axes unit vectors
$\hat{1}b, \hat{2}b, \hat{3}b$ = body principal axes unit vectors
I_1, I_2, I_3 = body moments of inertia
J_{11}, J_{12}, J_{13} = wheel number one moments of inertia about the $\hat{1}b, \hat{2}b, \hat{3}b$ axes, respectively
J_{31}, J_{32}, J_{33} = wheel number three moments of inertia about $\hat{1}b, \hat{2}b, \hat{3}b$ axes, respectively
ψ_1, ψ_3 = wheel rotation angles
$\omega_1, \omega_2, \omega_3$ = body angular rates with respect to inertial space about $\hat{1}b, \hat{2}b, \hat{3}b$ axes
$\hat{R}^c$ = unit vector from satellite center of mass to the center of attraction
I_1' = $I_1 + J_{31}$
I_2' = $I_2 + J_{12} + J_{32}$
I_3' = $I_3 + J_{13}$
k_1 = $(I_3' - I_2' + J_{33})/I_1'$ = body shape parameter
k_2 = $(I_3' - I_1' + J_{33} - J_{11})/I_2'$ = body shape parameter
k_3 = $(I_2' - I_1' - J_{11})/I_3'$ = body shape parameter
K_1, K_3 = spring resonant frequencies
η_1, η_2 = damping constants
r_1 = J_{11}/I_1'
r_3 = J_{33}/I_3'
ν_1 = J_{33}/I_1'
ν_2 = J_{11}/I_2'
ν_3 = J_{33}/I_2'
ν_4 = J_{11}/I_3'
g_1 = $-(k_1/2)\sin^2\phi\sin 2\theta$
g_2 = $(k_2/2)\sin\theta\sin 2\phi$
g_3 = $(k_3/2)\cos\theta\sin 2\phi$
h_1 = $-(k_1/2)\sin\theta\sin 2\phi$
h_2 = $-(k_2/2)\sin^2\theta\sin 2\phi$
h_3 = $-(k_3/2)[\sin^2\phi\cos^2\theta - \cos^2\phi]$
γ = body pitch angle
λ = parameter (greater than one) used in perturbation series
ω = $\gamma'/\lambda > 1$
η = $(1 + r_3)\eta_3$ = damping parameter
ω_0 = $(1 + r_3)^{1/2}K_3$ = natural frequency
ψ = ψ_3 = wheel angle of rotation
k = $\frac{3}{2}k_3$ = body shape parameter
r = $r_3/(1 + r_3)$ = inertia ratio
Ω = ψ' = wheel angular rate
H = Hamiltonian for pitch motion
H_0 = initial value of H
γ_c, γ_c' = variables evaluated at τ_c, the instant of capture
$\bar{\Omega}, \bar{\omega}, \bar{\psi}, \bar{\gamma}$ = long period or "average" value of $\Omega, \omega, \psi, \gamma$
$\tilde{\Omega}, \tilde{\omega}, \tilde{\psi}, \tilde{\gamma}$ = refinement in approximation of $\Omega, \omega, \psi, \gamma$
α, β, D = defined in Eq. (7)
N = $\lambda\bar{\omega}$ = average pitch tumbling rate
N_0 = initial value of N
N_c = value of N at capture
$(\tau_c)_{\min}$ = minimum time to go from $\tau = 0$ to $\tau = \tau_c$ with a given N_0
$\eta_{\text{opt}}, (\omega_0^2)_{\text{opt}}$ = optimum values of η, ω_0^2 giving $\tau = (\tau_c)_{\min}$

Received October 21, 1964; revision received February 23, 1965. This work was performed in association with research sponsored by NASA under Research Grant No. NsG-133-61. Parts of this research were included in Ref. 3.

* Scientist Associate/Research; formerly Research Assistant, Stanford University, Stanford, Calif. Member AIAA.

$\bar{\omega}_1, \bar{\omega}_2$ = average values of ω_1, ω_2
$\alpha_1, \alpha_2, \alpha_3, \alpha_4$ = amplitude of sinusoidal variations of ω_1, ω_2
$\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4$ = defined in Eq. (A4) of the Appendix
$\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4$ = defined in Eq. (A4)
$\bar{\theta}, \bar{\phi}$ = average values of θ, ϕ
$\bar{\psi}_1, a_1, b_1$ = defined in Eq. (A5)
$F_R, F_I, \mathfrak{D}$ = defined in Eq. (A5)
ϕ_0, θ_0 = initial value of ϕ, θ
R = defined in Eq. (17)
$J(N), P, Q$ = defined in Eq. (18)
C = coefficient representing a constant pitch torque of magnitude $n^2 I_3' C$
ω_s = frequency of synchronization (Fig. 6)
$\bar{P}$ = average pitch angular momentum

A. Introduction

THIS paper introduces a class of problems involving the damping of the tumbling motions of passively damped, gravity stabilized, artificial satellites. Simple examples of these problems are treated to indicate a general approach to the calculation of the motions and decay rate using perturbation theory. The use of perturbation methods may be necessary because of the extremely long intervals of the solutions in comparison with the length of integration steps necessary for a digital integration of the equations of motion. Analog computation methods are of little use for the same reasons; i.e., the long times of solution necessary to obtain the desired solutions allow the amplifiers to drift, and this causes large errors.

The system to be studied is a satellite with a tumbling rate of one or more times orbit rate which is to be damped by internal friction. That is, the effect of the gravity torques on the system is to disturb internal moving parts and thus dissipate energy, causing a gradual slowing of the tumbling until the satellite is captured into the libration region.

If one considers an orbiting system of particles with internal damping, and if one neglects the gravity and other external torques, he may use the law of conservation of angular momentum to show that, after a period of transient decay of the relative motion of particles, the general motion will be a tumbling motion with no relative motion between particles. The presence of gravity torques, however, invalidates the momentum conservation and produces ever-present relative motion between particles and thus ever-present friction. This friction damps the tumbling motion; systems with internal friction cannot remain tumbling if only acted upon by gravity torques and forces. If the tumbling rate is high as compared to orbit rate, the kinetic energy is large as compared to changes in the potential energy of gravity, and the tumbling motion is only slightly affected over an orbit period. This explains the slow attenuation of the tumbling for high initial rates.

This problem is of interest to engineers, because passively damped, gravity stabilized satellites may have initial conditions corresponding to tumbling motion, when they are released from the booster rocket or when they are struck by a meteorite. The problem also has interest as an application of the theory of perturbations, i.e., the method of averaging.[1,2]

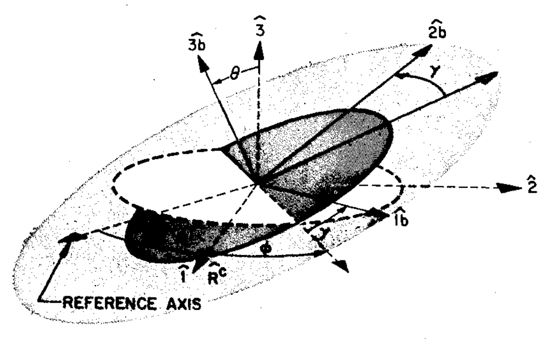

Fig. 1 Coordinate system.

A solution is given for a simple problem of the class just described, namely, a satellite tumbling in a gravitational field with two internally mounted inertia wheels arranged to rotate about the principal axis of tumbling (largest body moment of inertia) and about a transverse principal axis. These inertia wheels are restrained to the satellite body by means of springs and dashpots. The system using inertia wheels is only an idealized example to demonstrate the analytical methods. These methods can be applied to systems using gyros or other damping devices. The motions for such a system are obtained by use of a perturbation technique.

These motions can be separated into three phases: 1) rotation about the principal axis of maximum moment of inertia, with the axis of tumbling not aligned along the normal to the orbit plane, 2) convergence of the axis of tumbling to coincidence with the normal to the orbit plane, and 3) decay of the tumbling motion in the pitch (orbit) plane, until the occurrence of capture into the region of bounded libration. This sequence is only valid for initial tumbling rates not near the capture conditions. The time for the motion to converge to the pitch plane is quite short, and therefore the major emphasis in the analysis is directed toward the approximation to the pitch plane motion.

It is shown that optimum values of the damping and spring constants exist to give minimum time for the pitch plane tumbling to go from the initial angular velocity to the angular velocity of capture. For the system described, the time to capture increases sharply with increasing initial angular velocity.

By addition to the spring-mass-damping system of a certain small, constant pitch torque, e.g., by gas jets, we may cause the satellite angular velocity of tumbling to "synchronize" at a certain angular velocity, which is largely determined by the "resonance" frequency of the spring and sphere inertia. In this way, one may regulate the tumbling rate without complex sensing equipment. The calculation done, in connection with the "synchronization" effect, also serves to evaluate the final tumbling rate in the case of gas leakage.

The results discussed previously are obtained by using the asymptotic methods of approximation attributed to N. Krylov, N. Bogoliubov, and Y. A. Mitropolsky.[1,2] The method of averaging is used to eliminate the rapid, but predictable oscillatory motions excited by the gravity torque. The end result of employing the averaging method is a set of nonlinear differential equations for the averaged motion; the integration of these equations can be accomplished with much less effort than that necessary to integrate the complete equations of motion. If we use a digital computer, the integrations can be accomplished in fewer steps than a direct integration of the equations of motion.

B. Equations of Motion

Let us consider a rigid body moving in the orbit plane about a spherical attracting body. Its attitude is described by Euler angles ϕ, θ, γ relative to a rotating coordinate system $\hat{1}$ (unit vector along the radius vector), $\hat{3}$ (unit vector normal to the orbit plane), and $\hat{2}$ (unit vector along the velocity vector, such that $\hat{3} \times \hat{1} = \hat{2}$). Figure 1 shows the rotations from $\hat{1}$, $\hat{2}$, $\hat{3}$ to $\hat{1b}$, $\hat{2b}$, $\hat{3b}$ (the body frame) where, initially, $\hat{1b} = 1$, $\hat{2b} = \hat{2}$, and $\hat{3b} = \hat{3}$. To arrive at the new coordinates of the body ($\hat{1b}$, $\hat{2b}$, $\hat{3b}$), first rotate counterclockwise about $\hat{3}$ through an angle ϕ, then about $\hat{1b}$ counterclockwise through an angle θ, and finally rotate counterclockwise about $\hat{3b}$ through an angle γ.

The arrangement of two body-mounted inertia wheels is shown in Fig. 2. Each wheel is free to move about its axis of symmetry and is restrained by spring-damper torques $-J_{11}n^2(\eta_1\psi_1' + K_1^2\psi_1)$ about $\hat{1}b$ and $-J_{33}n^2(\eta_3\psi_3' + K_3^2\psi_3)$ about $\hat{3}b$. There is no wheel along axis $\hat{2}b$.

It is straightforward to derive the equations of motion of this system, taking account of the gravity torques acting on the vehicle. The equations are as follows:

Euler's Dynamical Equations

$$\left.\begin{aligned}
\omega_1' &= -k_1\omega_2\omega_3 + r_1(\eta_1\psi_1' + K_1^2\psi_1) - \\
&\qquad \nu_1\psi_3'\omega_2 + 3(g_1 \cos\gamma + h_1 \sin\gamma) \\
\omega_2' &= k_2\omega_1\omega_3 - \nu_2\psi_1'\omega_3 + \nu_3\psi_3'\omega_1 - \\
&\qquad 3(g_2 \cos\gamma + h_2 \sin\gamma) \\
\omega_3' &= -k_3\omega_1\omega_2 + r_3(\eta_3\psi_3' + K_3^2\psi_3) + \\
&\qquad \nu_4\psi_1'\omega_2 - 3(g_3 \cos2\gamma + h_3 \sin2\gamma)
\end{aligned}\right\} \quad (1)$$

Euler's Kinematical Equations

$$\left.\begin{aligned}
\theta' &= \omega_1 \cos\gamma - \omega_2 \sin\gamma \\
\phi' &= -1 + (1/\sin\theta)[\omega_1 \sin\gamma + \omega_2 \cos\gamma] \\
\gamma' &= \omega_3 - \cos\theta[\omega_1 \sin\gamma + \omega_2 \cos\gamma]
\end{aligned}\right\} \quad (2)$$

Equations of the Wheels

$$\left.\begin{aligned}
\psi_1'' + (\eta_1\psi_1' + K_1^2\psi_1) + \omega_1' &= 0 \\
\psi_3'' + (\eta_3\psi_3' + K_3^2\psi_3) + \omega_3' &= 0
\end{aligned}\right\} \quad (3)$$

where the wheels are assumed symmetrical about the axis of rotation.

The preceding differential equations are nonlinear and of tenth order. Physical reasoning and the computer simulations of A. E. Sabroff of Space Technology Laboratories show that, for $(\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2} \gg 1$, initially, the roll-yaw (out-of-orbit plane) motions ($\theta \neq 0$) decay rapidly, leaving the satellite tumbling only in the pitch (orbit) plane. Two separate analyses are made to check these results and give analytical answers.

1) The average tumbling angular rate N (expressed in multiples of the orbit rate, n) is allowed to vary slowly, and the motion of θ is examined.

2) The motion of the average angular rate N is calculated, assuming that the satellite rotates in the pitch plane only ($\theta = 0$). The capture problem is discussed to determine the behavior after N has decreased to the order of $N = 1$.

C. Tumbling Motion in the Pitch Plane

With the analysis given in Sec. D, we shall show that $\theta \rightarrow 0$ rapidly. Most of the time the body is simply tumbling in pitch. The differential equations of the pitch motion (assume $\theta = \psi_1 = \omega_1 = \omega_2 = \phi \equiv 0$) are (in first-order form)

$$\begin{aligned}
\gamma' &= \lambda\omega \qquad & \omega' &= (1/\lambda)[\eta r\Omega + r\omega_0^2\psi - k \sin2\gamma] \\
\psi' &= \Omega \qquad & \Omega' &= -\eta\Omega - \omega_0^2\psi + k \sin2\gamma
\end{aligned} \quad (4)$$

where

γ = body pitch angle
ψ_3 = ψ = wheel angle
η = $(1 + r_3)\eta_3$ = damping constant
r = $r_3/(1 + r_3) = J_{33}/(I_3' + J_{33})$ = inertia ratio
ω_0^2 = $K_3^2(1 + r_3)$ = spring natural frequency
k = $(\frac{3}{2})k_3$ = inertia parameter

We have introduced a new constant λ and a new variable ω by defining as $\lambda\omega \triangleq \gamma'$ the normalized tumbling rate. Furthermore, λ is defined to be equal to the final (capture) angular velocity, so that $\omega > 1$, always. $1/\lambda$ will be used as a small parameter in perturbation series.

1. Capture

From Eqs. (4), we can write the Hamiltonian H, an energy-like quantity, which is useful in solving the capture problem and giving limits of validity to the perturbation series expansions. This function and its derivative are

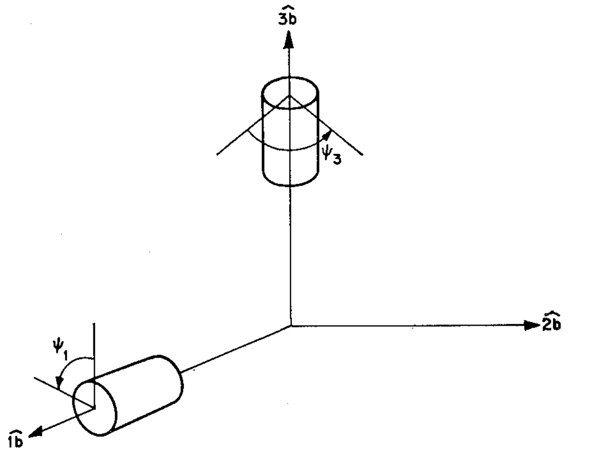

Fig. 2 Wheel location.

$$\begin{aligned}
H &= \tfrac{1}{2}[(\gamma')^2 + 2r\gamma'\Omega + r\Omega^2] + \\
&\qquad \tfrac{1}{2}[\omega_0^2 r/(1 + r)]\psi^2 + (1 - r)k \sin^2\gamma \\
H' &= -\eta r\Omega^2 < 0 \qquad (\eta > 0)
\end{aligned} \quad (5)$$

The integration of the second of Eqs. (5) leads to the inequality

$$H \leq H_0 \quad (6)$$

where H_0 is H of Eqs. (5) evaluated at the initial instant.

Physically H' is negative, and H decreases during the tumbling motion. When H reaches a value $(1 - r)k$, then the subsequent motion will be bounded. This means that in less than one tumble capture will occur.

If the inertia parameter $J_{33}/(I_3' + J_{33}) = r \ll 1$, then the equation of energy at capture is

$$\tfrac{1}{2}\gamma_c'^2 = (1 - r)k \cos^2\gamma_c$$

where γ_c, γ_c' are γ, γ' evaluated at capture. This, by analogy with a simple pendulum, is the largest closed energy contour surrounding the equilibrium points $\gamma = 0, \pm\pi, \pm2\pi, \ldots$ The capture angular velocity is, therefore,

$$\gamma_c' = \pm[2k(1 - r)]^{1/2} \cos\gamma_c$$

If $\cos\gamma_c = 1$, and the motion reaches $\gamma_c' = \pm[2k(1 - r)]^{1/2}$, then in one revolution it will surely be captured because of the energy loss. The preceding arguments lead to a choice of $\lambda = [2k(1 - r)]^{1/2}$ for r small, $\gamma' > 0$, $\omega_c > 1$.

2. Perturbation Solution for the Tumbling Motion

Approximate solutions will be obtained when the angular velocity $\gamma' = \lambda\omega$ is large as compared to one. Actually, the results will be approximately correct anywhere out of the capture region. We use $1/\lambda$ as a small parameter in the expansions, assuming that $\omega > 1$. If one uses the averaging method,[1,2] he obtains series solutions in powers of $1/\lambda$, and can thus approximate the actual motion. To begin, we must get the equations in "standard form" by a transformation. We use, for this purpose, the forced solution with constant ω. The reasoning is that for large λ, $\omega' \approx 0$ and we have just a sinusoidally forced set of equations. The transformation becomes

$$\left.\begin{aligned}
\Omega &= \bar{\Omega} + 2\lambda\bar{\omega}[-\alpha \sin2\bar{\gamma} + \beta \cos2\bar{\gamma}] + \tilde{\Omega} \\
\omega &= \bar{\omega} + (1/2\lambda^2\bar{\omega})[k \cos2\bar{\gamma} + r\omega_0^2(\alpha \sin2\bar{\gamma} - \\
&\qquad \beta \cos2\bar{\gamma}) + 2\lambda\omega r\eta(\alpha \cos2\bar{\gamma} + \beta \sin2\bar{\gamma})] + \tilde{\omega} \\
\psi &= \bar{\psi} + (\alpha \cos2\bar{\gamma} + \beta \sin2\bar{\gamma}) + \tilde{\psi} \\
\gamma &= \gamma + (1/4\lambda^2\bar{\omega}^2)[k \sin2\bar{\gamma} - r\omega_0^2(\alpha \cos2\bar{\gamma} + \\
&\qquad \beta \sin2\bar{\gamma}) + 2\lambda\bar{\omega}r\eta(\alpha \sin2\bar{\gamma} - \beta \cos2\bar{\gamma})] + \tilde{\gamma}
\end{aligned}\right\} \quad (7)$$

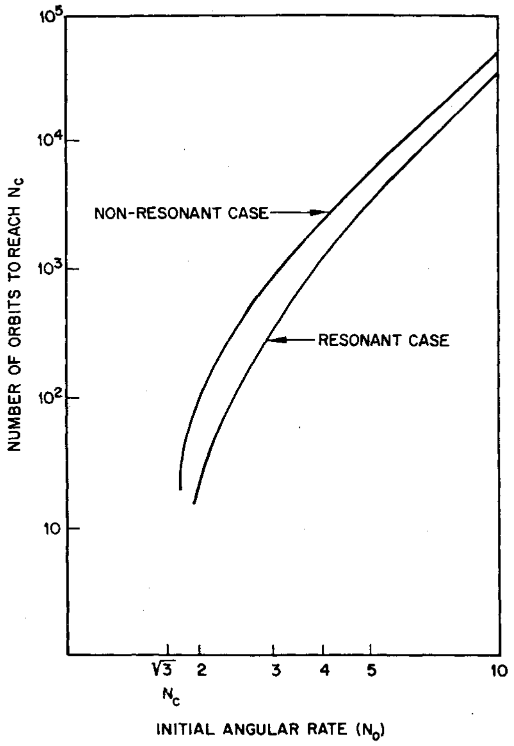

Fig. 3 Damping times for varying initial rates: optimum spring and damping constants $K = 0.01$.

where

$$\alpha = -\frac{D\eta}{2\lambda\bar{\omega}} \qquad \beta = \frac{D(\omega_0^2 - 4\lambda^2\bar{\omega}^2)}{4\lambda^2\bar{\omega}^2}$$

$$D = \frac{k}{\eta^2 + [(\omega_0^2 - 4\lambda^2\bar{\omega}^2)/2\lambda\bar{\omega}]^2}$$

Here the barred variables represent the "secular" terms caused by damping, and the $\sin 2\bar{\gamma}$, $\cos 2\bar{\gamma}$ terms are the forced response due to gravity, for large λ. The $\tilde{\gamma}$, $\tilde{\omega}$, $\tilde{\Omega}$, $\tilde{\psi}$ variables represent additional perturbations to be defined later. They will be set equal to zero for the present.

The preceding equations are to be viewed as a transformation from Ω, ω, γ, ψ variables to the barred variables $\bar{\Omega}$, $\bar{\omega}$, $\bar{\gamma}$, $\bar{\psi}$. The oscillating terms in $2\bar{\gamma}$ are the result of using the $\sin 2\bar{\gamma}$ terms in Eqs. (4) to force the equations in a periodic oscillation (as if $\lambda\bar{\omega}$ were a constant). It must be emphasized that no approximation is implicit in Eqs. (7), but merely a transformation based on an approximate way of viewing Eqs. (4).

Now we are going to use Eqs. (7) to get the differential equations (4) in terms of the barred variables. By direct substitution of Eqs. (7) into Eqs. (4) (with the $\tilde{\Omega}$, $\tilde{\omega}$, $\tilde{\gamma}$, $\tilde{\psi}$ set equal to zero), the following differential equations in the barred variables are obtained carrying terms up to fourth order in $1/\lambda$:

$$\left.\begin{aligned}
\bar{\gamma}' &= \lambda\bar{\omega}[1 - (k/2\lambda^2\bar{\omega}^2)\cos 2\bar{\gamma}] + 0(1/\lambda^4) \\
\lambda\bar{\omega}' &= r(\eta\bar{\Omega} + \omega_0^2\bar{\psi}) - (k/4\lambda^2\bar{\omega}^2)\{(k/2)\sin 4\bar{\gamma} + \\
&\quad r\omega_0^2(-\alpha)(1 + \cos 4\bar{\gamma}) + (r\omega_0^2/2)(-\beta)\sin 4\bar{\gamma} + \\
&\quad \lambda\bar{\omega}r\eta\alpha\sin 4\bar{\gamma} - \lambda\bar{\omega}r\eta\beta(1 + \cos 4\bar{\gamma})\} + 0(1/\lambda^4) \\
\bar{\Omega}' &= -(\eta\bar{\Omega} + \omega_0^2\bar{\psi}) + (k/4\lambda^2\bar{\omega}^2)\{k\sin 4\bar{\gamma} + \\
&\quad r\omega_0^2[-\alpha(1 + \cos 4\bar{\gamma}) - \beta\sin 4\bar{\gamma}] + \\
&\quad 2\lambda\bar{\omega}r\eta[\alpha\sin 4\bar{\gamma} - \beta(1 + \cos 4\bar{\gamma})]\} + 0(1/\lambda^4) \\
\bar{\psi}' &= \bar{\Omega} + (k/2\lambda\bar{\omega})[-\alpha\sin 4\bar{\gamma} + \beta(1 + \\
&\quad \cos 4\bar{\gamma})] + 0(1/\lambda^4)
\end{aligned}\right\} \quad (8)$$

Equations (8) are the Eqs. (4) in terms of the barred variables, and these differential equations possess the "standard form" of Bogoluibov and Mitropolsky.[1,2] That is to say, the right-hand sides are small, and thus the barred variables vary slowly; the averaged differential equation may then be solved to find the secular variations in the barred variables. The averaged (secular) equations in the barred variables are

$$\left.\begin{aligned}
\bar{\gamma}' &\triangleq N = \lambda\bar{\omega} \\
N' &= \lambda\bar{\omega}' = r(\eta\bar{\Omega} + \omega_0^2\bar{\psi}) - \\
&\qquad (kr/4N^2)[-\omega_0^2\alpha - 2N\eta\beta] \\
\bar{\Omega}' &= -(\eta\bar{\Omega} + \omega_0^2\bar{\psi}) - (kr/4N^2)(\omega_0^2\alpha + 2N\eta\beta) \\
\bar{\psi}' &= \bar{\Omega} + (k\beta/2N)
\end{aligned}\right\} \quad (9)$$

Based on the transformed variables as defined by Eqs. (9), we may write the approximate solutions including the effects of the oscillating terms in Eqs. (8). These are

$$\left.\begin{aligned}
\tilde{\gamma} &= -(k/4N^2)\sin 2\bar{\gamma} \qquad \lambda\tilde{\omega} = -(k^2/32N^3)\cos 4\bar{\gamma} \\
\tilde{\Omega} &= -(k/16N^3)\{k\cos 4\bar{\gamma} + r(\omega_0^2\alpha + \\
&\qquad 2N\eta\beta)\sin 4\bar{\gamma} - r(\omega_0^2\beta - 2N\eta\alpha)\cos 4\bar{\gamma}\} \\
\tilde{\psi} &= -(k/8N^2)[\alpha\cos 4\bar{\gamma} - \beta\sin 4\bar{\gamma}]
\end{aligned}\right\} \quad (10)$$

These expressions [Eqs. (10)] are of order $1/\lambda^2$ or smaller. The barred variables are solutions to the averaged Eqs. (9), and together with Eqs. (7) and (10) they comprise the approximate solutions to Eqs. (4) up to order $(1/\lambda^2)$. The differential equations of average motion [Eqs. (9)] are all that remain to be solved. These are solved analytically in the next section. In a more complicated case, these would have to be integrated on a digital computer. This could be done with many fewer steps of integration than a pointwise integration of the original equations of motion, because the velocity and position variables are smoothed by the averaging process.

An investigation of the errors in these perturbation calculations for $\omega > 1$, $\lambda = [2k(1 - r)]^{1/2}$ shows good convergence if the inertia parameter $r < 0.1$ regardless of the value of the damping η. The dominant oscillatory term in the expression for ω is of amplitude $1/[4(1 - r)]$, which is certainly smaller than $\bar{\omega} > 1$ for $r < 0.1$. The next term in the expression for ω is $\eta r/2(2)^{1/2}$ for $\eta < 1$, or $(k^{1/2}r)/\eta(2)^{1/2}$ for $\eta \gg 1$. These terms are quite small as compared to $\bar{\omega} > 1$. The amplitude of $\tilde{\omega}$ is $\frac{1}{64}$ at capture. These estimates of the oscillatory terms show good convergence for the stated conditions. The conditions $r < 0.1$ and $\lambda = [2k(1 - r)]^{1/2}$ are quite realistic.

3. Solution of the Secular Equations

Secular equations (9) remain to be solved for the average behavior of the variables. If we assume that the transient oscillation of the average wheel position $\bar{\psi}$ has decayed rapidly, and that only the "steady-state" value of $\bar{\psi}$, $\bar{\Omega}$ remains ($\bar{\psi}' = \bar{\Omega}' = 0$), then

$$(\eta\bar{\Omega} + \omega_0^2\bar{\psi}) = -(kr/4N^2)[\omega_0^2\alpha + 2N\eta\beta]$$

$$\bar{\Omega} = -(k\beta/2N)$$

This makes the equation for the average tumbling rate N

$$N' = -[(1 - r)r\eta Dk/2N] \quad (11)$$

The solution of Eq. (11) by separation of variables gives

$$\tau_c = \frac{1}{k^2(1 - r)r\eta}\left[\eta^2(N_0^2 - N_c^2) + 2(N_0^4 - N_c^4) + \frac{\omega_0^4}{2}\log_e\left(\frac{N_0}{N_c}\right) - 2\omega_0^2(N_0^2 - N_c^2)\right] \quad (12)$$

where N_0 is the initial angular rate of tumbling, and N_c is the rate at time of capture $\tau = \tau_c$.

Notice that in Eq. (12) τ_c is independent of the sign of N_0. The minimum value of τ_c occurs when the derivatives of τ_c with respect to η and ω_0^2 vanish. This occurs for the following optimum values of the damping and spring resonance frequencies:

$$(\omega_0^2)_{opt} = 2(N_0^2 - N_c^2)/\log_e(N_0/N_c)$$
$$\eta_{opt} = 2\{N_0^2 + N_c^2 - [(\omega_0^2)_{opt}/2]\} \tag{13}$$

The minimum value of the capture time is given from Eqs. (12) and (13) as

$$(\tau_c)_{min} = [2(N_0^2 - N_c^2)/k^2(1-r)r]\eta_{opt} \tag{14}$$

It is useful to compare the minimum value of τ_c with the value for arbitrary values of damping and spring constant; this gives the ratio

$$\frac{\tau_c}{(\tau_c)_{min}} = \frac{1}{2}\left[\frac{\eta}{\eta_{opt}} + \frac{\eta_{opt}}{\eta} + \frac{(\omega_0^2)_{opt}}{\eta\eta_{opt}}\left(\frac{\omega_0^2}{(\omega_0^2)_{opt}} - 1\right)^2\right] \tag{15}$$

Table 1 gives the computed values of the optimum capture time for a particular value of terminal angular velocity.

If the spring is removed, then $\omega_0^2 = (\omega_0^2)_{opt} = 0$, and the optimum η for this "nonresonant" case can be calculated from Eq. (13). Figure 3 shows the relative performance of "optimum" systems for the "resonant" case ($\omega_0 \neq 0$) and the "nonresonant" case ($\omega_0 = 0$). The resonant oscillation behavior caused by the spring is seen to decrease the damping time $(\tau_c)_{opt}$; in both cases, the damping time increases with increasing N_0 sharply from $N_0 = N_c$ until, for $N_0 = 3$, it increases approximately as the cube of N_0 (Fig. 3).

This behavior may be justified on physical grounds. The "resonance" phenomenon causes larger relative motion between the sphere and the satellite body, thus dissipating more energy per orbit and damping γ' more rapidly (Fig. 4). The rapid increase of damping time with increasing N_0 is, of course, the reason for the perturbation technique; this is a direct result of the "internal" nature of the damping.

If there were no gravity, one could immediately use the angular momentum principle to show that the particles of the system are frozen to some rotating reference axes in steady state; there would be no relative motion between particles, and thus no internal dissipation of energy. If there is gravity torque, then one has relative particle motion and thus damping. Notice, however, that this relative motion is very small for large rotation velocity, because the kinetic energy of rotation is much larger than the fluctuations in potential energy due to gravity.

In most practical systems, there exist limit stops on the motion of the wheels or gyros used for damping. These stops make the equations highly nonlinear. The main effect, however, is a "jump phenomenon" near resonance of ψ motion. This can be calculated by the method of averaging, and it can be shown to lengthen the response time.

Figure 5 illustrates how the effect of gravity on the angular velocity increases as the motion nears capture; the effect is to increase the "ripples" of oscillation about the average angular velocity N. This method of perturbations saves computer integration over all these rapidly varying functions; integration of functions varying as the average velocity N can be accomplished with many fewer computer steps.

Table 1 Optimum time to capture

$K = k^2(1-r)r$
$N_c = (3)^{1/2}$

N_0	η_{opt}	$(\omega_0^2)_{opt}$	$K\times$ (No. of orbits to N_c)	No. of orbits ($K = 0.01$)
1.8	0.482	12.25	0.037	3.7
1.9	0.790	12.60	0.154	15.0
2.0	1.025	12.96	0.327	33.0
3.0	1.414	22.00	2.690	270.0
4.0	3.640	31.00	15.100	1,510.0
5.0	3.800	41.50	26.600	2,660.0
10.0	9.750	110.50	302.000	30,200.0

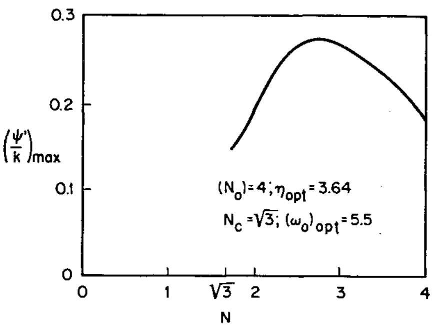

Fig. 4 Motion of ψ'_{max} for optimum resonant damping from $N_0 = 4$.

D. Motions for $\theta \neq 0$

When the satellite, spinning about its axis of greatest inertia, is released, it may be oriented with its spin axis $\hat{3}b$ not coincident with the normal to the orbit plane $\hat{3}$. Thus, the angle θ is nonzero. The equation of secular motion of θ ($\bar{\theta}$ is the average value of θ) can be shown to be (see Appendix)

$$\frac{d\bar{\theta}}{d\tau} = -\frac{3r_1\eta_1k_1(1-k_2)N^2\mathfrak{D}}{8(1-k_1k_2)}\sin 2\bar{\theta} \tag{16}$$

where

$$\mathfrak{D} = 1/\{[K_1^2(1+r_1) - N^2]^2 + [(1+r_1)^2\eta_1^2N^2]\}$$

This equation can be integrated to find the motion of $\bar{\theta}$; such integration demands a knowledge of $N(\tau)$ as given in the preceding section. Let us transform to $N(\tau)$ as a convenient independent variable. Using Eq. (11) gives

$$d\bar{\theta}/dN = kR(N^3\mathfrak{D}/D)\sin 2\bar{\theta} \tag{17}$$

where

$$R = \frac{3}{4}\frac{\eta_1}{\eta}\frac{r_1k_1(1-k_2)}{r(1-r)k^2(1-k_1k_2)}$$

The solution to Eq. (17) is found by direct integration:

$$\tan\bar{\theta}/\tan\bar{\theta}_0 = e^{J(N)} \tag{18}$$

where

$$J(N) = 2R\int_{N_0^2}^{N^2}\frac{P(x)}{Q(x)}dx$$

$$P(x) = [4\eta^2x + (\omega_0^2 - 4x)^2]$$

$$Q(x) = [K_1^2(1+r_1) - x]^2 + \eta_1^2(1+r_1)^2x$$

The integral $J(N)$ is always negative ($N < N_0$) and, therefore, $\tan\bar{\theta} \to 0$ and $\bar{\theta} \to \pm n\pi$ ($n = 0, 1, 2, \ldots$), as N decreases. Notice that if

$$N_0 > \max[(\omega_0/2), K_1(1+r_1)^{1/2}]$$

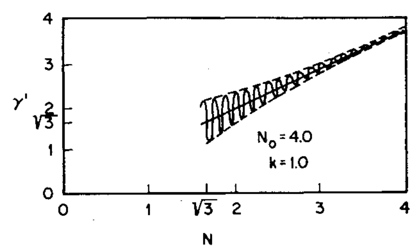

Fig. 5 Motion of the tumbling rate from $N_0 = 4$ (secular plus periodic); resonant case.

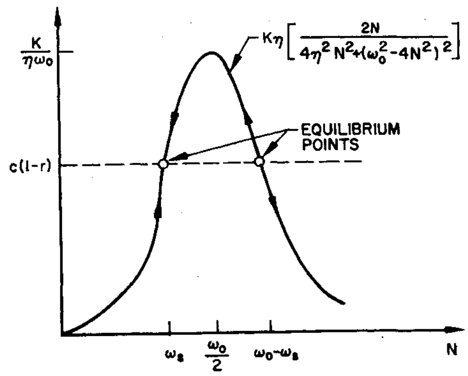

Fig. 6 Location of equilibrium points with constant thrust.

then, approximately, $J(N) = 32R(N^2 - N_0^2)$. If R is the order of one, then for $\tan\bar{\theta}/\tan\bar{\theta}_0 \leq 1/e^3$, $N_0^2 - N^2 \approx \frac{1}{10}$. If $\Delta N = N - N_0$, then $\Delta N \approx \frac{1}{20} N_0$. This means that, even for a very high initial tumbling rate of $N_0 = 5$, the motion only lasts for a time equivalent to $N = 0.01$. Thus, the percentage of time spent tumbling with $\theta \neq 0$ is small. Equation (18) may be used to evaluate quantitatively the effect of $\theta \neq 0$ motion for given N_0 and given parameters.

It should be observed that the approximation implicit in Eq. (16), i.e., the average tumbling rate $N \gg 1$, becomes less valid as capture is approached. It may well be that the pitch plane tumbling motion becomes unstable because of parametric excitation of roll-yaw resonances. This has not been investigated, but it should not invalidate the calculated motion of pitch tumbling. This is true because the difficulty arises in a relatively small interval of time around capture.

It is important to observe that the presence of limit stops on damping devices may introduce new complications into the analysis (see Sec. C). The vehicle may capture while out of the pitch plane if the initial tumbling angular velocity is too low; this case is excluded from the previous analysis.

E. Synchronization of the Tumbling Rate

If a small, constant pitch torque (for example, via gas jets or gas leakage) of magnitude $n^2I_3'c$ is added, an interesting phenomenon occurs. The average speed N of the main satellite body rises from zero to a certain value determined by a parameter ω_s, and then the satellite remains turning at a constant average speed. If the initial speed is too large, this does not occur; the body simply spins up indefinitely.† The equation of average motion is, with the additional torque,

$$N' = -K\eta\{2N/[4\eta^2N^2 + (\omega_0^2 - 4N^2)^2]\} + (1 - r)c \quad (19)$$

where we have defined $K = (1 - r)rk^2$.

There are, of course, equilibrium points at which $N' = 0$. These are the points of synchronization that are sought. They are solutions to Eq. (19) with $N' = 0$. This can be seen graphically by noticing Fig. 6, which shows the solution to $N' = 0$. From this solution we may deduce that $N = \omega_s$ is a stable point, and $N = \omega_0 - \omega_s$ is unstable. The arrows show the transient motions. $N = \omega_0 - \omega_s$ is the maximum initial speed for stable behavior.

F. Conclusions

Several problems related to tumbling satellites have been discussed. These problems pose some interesting questions that involve the length of time it takes to damp a satellite with only internal damping between the moving parts. It is instructive to notice the value of the average rate of change of angular momentum. Letting P denote the total pitch angular momentum we have

$$\bar{P} = N + r\bar{\Omega}$$

$$\bar{P}' = -(1 - r)r(\eta kD/2N) = r0(1/N^3) \quad (20)$$

where $\bar{P}$ denotes the average pitch angular momentum (divided by $J_{33} + I_3'$). The important observation about Eqs. (20) is that the average torque (or rate of change of total angular momentum) is small (of order $1/N^3$). This fact indicates that the average torque about the pitch axis due to the damping torques between particles of the system is of order $1/N^3$.

Because of tumbling rates significantly above the orbit rate, the angular momentum decreases as the cube of the reciprocal angular rate, and the decrease in tumbling rate is extremely slow. One must notice that this would not necessarily hold if external damping torques, for example, gas jets or earth-magnetic field damping, were significant. The slow nature of the secular decay of tumbling rate makes direct digital solution of the equations of motion very costly in time and in roundoff errors. The perturbation scheme outlined makes the job of integration much less difficult and in certain cases obviates the need for numerical integration entirely.

If the reasoning of this paper is applied to a spinning space station, where N may be 100 or more, the same convergence of the motions into the orbit plane may be shown to obtain. The energy losses would be caused by attitude control, and the "resonance" phenomena would occur at a frequency of the order of N.

† In this analysis, it is assumed that the internal damping is the only damping present. In case other sources are present, they may be easily accounted for.

Appendix: Out-of-Orbit Plane Motions

The yaw-roll motions ($\theta \neq 0$) will be discussed here under the assumption of a slowly varying average pitch tumbling rate N. The equations of motion given in Eqs. (1–3) can be written in simpler form for constant N:

$$\omega_1' = -k_1N\omega_2 + 3(g_1\cos\gamma + h_1\sin\gamma) + r_1(\eta_1\psi_1' + K_1^2\psi_1) \quad (A1)$$

$$\omega_2' = k_2N\omega_1 - 3(g_2\cos\gamma + h_2\sin\gamma)$$

$$\theta' = \omega_1\cos\gamma - \omega_2\sin\gamma$$

$$\phi' = -1 + (1/\sin\theta)\,[\omega_1\sin\gamma + \omega_2\cos\gamma] \quad (A2)$$

$$\gamma' = N$$

$$\psi_1'' + (1 + r_1)(\eta_1\psi_1' + K_1^2\psi_1) = k_1N\omega_2 - 3(g_1\cos\gamma + h_1\sin\gamma) \quad (A3)$$

where $\nu_1\psi_3' \ll k_1\omega_3$, $\nu_2\psi_1' \ll k_2\omega_1$, $\nu_3\psi_3' \ll k_2\omega_3$ are neglected. For practical cases, r_1, r_2 are both less than 0.1. Thus, we shall use approximations that are asymptotically correct if r_1, r_3 are small. First, let $r_1 = 0$ in Eq. (A1) and solve for ω_1, ω_2. This gives

$$\omega_1 = \bar{\omega}_1 + \alpha_1\cos\bar{\gamma} + \beta_1\sin\bar{\gamma}$$

$$\omega_2 = \bar{\omega}_2 + \alpha_2\cos\bar{\gamma} + \beta_2\sin\bar{\gamma} \quad (A4)$$

where $\alpha_1 = \bar{\alpha}_1 + r_1\bar{\bar{\alpha}}_1$, etc. Then putting Eq. (A4) into Eq. (A1) and solving gives

$$\bar{\alpha}_1 = -\frac{1}{1 - k_1k_2}\left[\frac{3h_1}{N} + \frac{3g_2}{N}k_1\right]$$

$$\bar{\alpha}_2 = -\frac{1}{1 - k_1k_2}\left[\frac{-3h_2}{N} + \frac{3g_1}{N}k_2\right]$$

$$\bar{\beta}_1 = -\frac{1}{1 - k_1k_2}\left[\frac{-3g_1}{N} + \frac{3h_2}{N}k_1\right]$$

$$\bar{\beta}_2 = -\frac{1}{1 - k_1k_2}\left[\frac{3g_2}{N} + \frac{3h_1}{N}k_2\right]$$

where the "slowly varying" variables $\bar{\theta}$, ϕ have been held constant.

Now, use Eq. (A4) in Eq. (A3) to obtain the forced response of ψ_1. Let

$$\psi_1 = \bar{\psi}_1 + a_1 \cos\gamma + b_1 \sin\gamma \tag{A5}$$

The coefficients are found to be, after some simple algebra,

$$a_1 = \{F_R[(1 + r_1)K_1^2 - N^2] + F_I(1 + r_1)\eta_1 N\}\mathfrak{D}$$

$$b_1 = \{-F_I[(1 + r_1)K_1^2 - N^2] + F_R(1 + r_1)\eta_1 N\}\mathfrak{D}$$

$$F_R = (-3g_1 + k_1 N\bar{\alpha}_2)$$

$$F_I = (3h_1 - k_1 N\bar{\beta}_2)$$

$$\mathfrak{D} = 1/\{[K_1^2(1 + r_1) - N^2]^2 + (1 + r_1)^2\eta_1^2 N^2\}$$

Using Eq. (A5) in the Euler equations [Eq. (A1)] to get the corrected values of α_1, α_2, β_1, β_2 to first order in r_1 gives

$$\tilde{\alpha}_1 = [(-1/N)/(1 - k_1k_2)][K_1^2 b_1 - \eta_1 N a_1]$$

$$\tilde{\alpha}_2 = [(-k_2/N)/(1 - k_1k_2)][\eta_1 N b_1 + K_1^2 a_1]$$

$$\tilde{\beta}_1 = [(1/N)/(1 - k_1k_2)][\eta_1 N b_1 + K_1^2 a_1]$$

$$\tilde{\beta}_2 = [(-k_2/N)/(1 - k_1k_2)][K_1^2 b_1 - \eta_1 N a_1]$$

We are now in a position to use the approximate motion of ω_1, ω_2 to find the average motion of ϕ, θ (or $\bar{\phi}$, $\bar{\theta}$, using bars to denote average values). Averaging Eq. (A2) gives for small $|\omega_1|$, $|\omega_2|$

$$\begin{aligned} \bar{\theta}' &= \tfrac{1}{2}(\alpha_1 - \beta_2) \\ \bar{\phi}' &= -1 + \tfrac{1}{2}(1/\sin\bar{\theta})(\beta_1 + \alpha_2) \end{aligned} \tag{A6}$$

It is easy to see from the definitions that α_1, α_2, β_1, β_2 are small of order $1/N$. This means that approximately $\bar{\phi} = -\tau + \phi_0$. We average over τ again and obtain "average" results for the $\bar{\theta}$ motion as

$$\bar{\theta}' = -[3r_1\eta_1 k_1(1 - k_2)N^2\mathfrak{D}/8(1 - k_1k_2)] \sin 2\bar{\theta} \tag{A7}$$

References

[1] Bogoliubov, N. N. and Mitropolsky, Y. A., *Asymptotic Methods in the Theory of Non-Linear Oscillations* (Hindustan Publishing Corp., Delhi, India, 1961), pp. 387–427.

[2] Minorsky, N., *Nonlinear Oscillations* (D. Van Nostrand, Inc., New York, 1962), pp. 273–281.

[3] Pringle, R., Jr., "On the capture, stability, and passive damping of artificial satellites," Ph.D. Thesis, Stanford Univ. (June 1964); also Stanford Univ., Dept. of Aeronautical Engineering, SUDAER Rept. 181 (1964).